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A New Method to Explore the Integer Partition Problem

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Abstract

Integer Partition is a well known difficult problem in number theory and no solid solution has been found until today. The authors used Linear Difference Equation and Root of Unity to successfully get the expressions for integer n when the group number is k=2, k=3, k=4 and k=5. This proves the close relationship between integer partition and root of unity. With the analysis, it can be concluded that there will be an expression of U(n,k) existing for any n and k and we have found a method to develop the expression. Also this has provided a new interesting method to do further research in the integer partition area.

Keywords

Integer Partition, Forward Difference Operator, Shift Operator, Linear Difference Equation, Root of Unity

1. The Introduction Of Integer Partition Problem

Integer partition is a fascinating category of mathematics with a long history and many applications. It has played a significant role in the development of number theory and has contributed to various fields such as computer science, physics, and music theory. The study of integer partition continues to be an active area of research, and it is likely to lead to many new discoveries in the future[1].

In number theory, a partition of a positive integer *n*, also known as an integer partition, is a way of writing *n* as a sum of positive integers. The goal of integer partition is to figure out the number of ways a number n can be broken down k ways. For example, 4 can be broken down as (1+1+1+1), (1+1+2), (2+2), (3+1), and (4). Repeats of combinations like (3,1) and (1,3) will not count[2].

Below is an example of the number of ways U(n,k) when n is from 1 to 10, k is from 1 to 10. (Blank spaces are equivalent to 0, when n<k)

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	(k) 1	2	3	4	5	6	7	8	9	10
(n) 1	1									
2	1	1								
3	1	1	1							
4	1	2	1	1						
5	1	2	2	1	1					
6	1	3	3	2	1	1				
7	1	3	4	3	2	1	1			
8	1	4	5	5	3	2	1	1		
9	1	4	7	6	5	3	2	1	1	
10	1	5	8	9	7	5	3	2	1	1

Figure 1. The value of U(n,k)

When n and k are very small, we can easily find the numbers but when n and k become very large, it is extremely hard to find the value of U(n,k). No closed formal expression for the partition problem is known till today, although there are some algorithms to do recursive computing via computer[3].

2. LINEAR DIFFERENCE EQUATIONS AND ROOT OF UNITY

The Difference Equation is an equation that shows the functional relationship between an independent variable and consecutive values or consecutive differences of the dependent variable.

To get the general solution for a Linear Difference Equation, we need to get the solution of the homogeneous equation first, then get the solution for the nonhomogeneous. The general solution is to combine the two solutions altogether. The final steps are to find the constant coefficients for each variable.

In mathematics, a root of unity, is any complex number that yields 1 when raised to some positive integer power *n*. For example, the roots of unity are the solutions to the equation $x^k = 1$, where n is a positive integer. In other words, the roots of unity are the complex numbers that satisfy the equation (cos $2\pi m/k$) + i(sin $2\pi m/k$) for m = 0, 1, 2, ..., k-1. These roots form a regular polygon in the complex plane, with the vertices located at the roots of unity.

We notice the root of unity has a close relationship with integer partition problems.
1. Do k partition on a large number n is very similar to cut a circle into k pieces
2. Both cos and sin has the periods no matter how large the number n is, so this will help us to get the results
3. We will use Linear Difference Equation to do the calculation, which has the roots to be associated with roots of unity

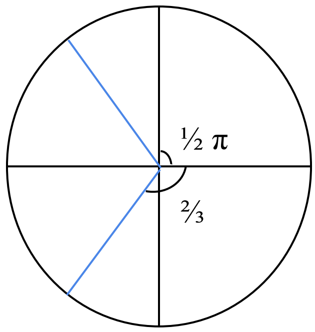

Figure 2. Root of unity

In this picture, a circle can be divided by 3 parts with an angle like $\frac{2}{3}\pi$

Or it can also be divided by 4 parts with an angle like $\frac{1}{2}\pi$

We all know that:

For 3 parts of a circle, there are 3 roots:

$$x_1 = 1,\ x_2 = \cos\frac{2\pi}{3} + i\sin\frac{2\pi}{3} = -\frac{1}{2} + \frac{\sqrt{3}}{2}i,$$

$$x_3 = \cos\frac{2\pi}{3} - i\sin\frac{2\pi}{3} = -\frac{1}{2} - \frac{\sqrt{3}}{2}i$$

For 4 parts of a circle. there are 4 roots:

$$x_1 = 1,\ x_2 = \cos\frac{\pi}{2} + i\sin\frac{\pi}{2} = i,$$

$$x_3 = \cos\frac{4\pi}{4} = -1,\ x_4 = \cos\frac{3\pi}{2} + i\sin\frac{3\pi}{2} = -i$$

for any k: there will be k roots as well.

$$x^k - 1 = (x-1)\left(x^{k-1} + x^{k-2} + \ldots + x + 1\right)$$

3. Get the Expressions for Integer Partition

We all know:

$$U(n,k) = U(n-1,k-1) + U(n-k,k) \text{ and}$$
$$U(n,k) = U(n-k,1) + U(n-k,2) + U(n-k,3) + \ldots + U(n-k,k)$$

Here, n stands for the whole number , and k stands for the group for partition. U(n,k) is the number for integer n when we do k partitions. A is defined as the constant coefficient. Then the Integer Partition problem is converted to find the solution for the nonhomogeneous linear difference equation.

We also have this definition:

$$n^{(1)} = n$$

$$n^{(2)} = n(n-1) = n^2 - n$$

… …

$$n^{(r)} = n(n-1)(n-2)\cdot\ \cdot\ \cdot\ (n-r+1) = \frac{n!}{(n-r)!}$$

When k=1, it is very simple:

$$Un = 1 = \cos 2n\pi$$

(1) For K=2

Here we use linear difference equation method to get the expressions

$$Un+2\ -\ Un\ =\ 1$$

$$\left(E^2\ -\ 1\right) Un\ =\ 1$$

$$Un = \ \frac{1}{(E+1)(E-1)}\ n^{(0)}$$

$E - 1 = \Delta$, here the symbol E is the shift operator. the symbol Δ is called the forward difference operator

$$Un\ =\ \frac{1}{(E+1)\Delta}\ n^{(0)}$$

$$Un\ =\ \frac{1}{E+1}\ n^{(1)}$$

$$Un\ =\ \frac{1}{2\left(1+\frac{1}{2}\Delta\right)}\ n^{(1)}$$

$$Un\ =\ \frac{1}{2}\left(1-\frac{1}{2}\Delta+\left(\frac{1}{2}\Delta\right)^2-\left(\frac{1}{2}\Delta\right)^3+\left(\frac{1}{2}\Delta\right)^4\dots\right)\ n^{(1)}$$

$$Un\ =\ \frac{1}{2}\left(n-\frac{1}{2}\right)\ =\ \frac{1}{2}n-\frac{1}{4}$$

Hence, the general solution is like this:

$$Un\ =\ \frac{1}{2}n-\frac{1}{4}+A_1(1)^n+A_2(-1)^n$$

While n=1, and n=2, we have 2 equations to help us determine the constant A1 and A2

$$\frac{1}{2}\ \times\ 2\ -\ \frac{1}{4}+A_1+A_2\ =\ 1$$

$$\frac{1}{2}\times 3-\frac{1}{4}+A_1-A_2=1$$

Then

$$A_1 = 0 \qquad A_2 = \frac{1}{4}$$

Then we have the general solution when k=2

$$\frac{1}{2}n - \frac{1}{4} + \frac{1}{4}(-1)^n$$

$$*(-1)^n = \cos n\pi$$

(2)For K=3,

$$Un + 3 - Un = \frac{1}{2}n - \frac{1}{4} + \frac{1}{4}(-1)^n + 1$$

$$(E^3 - 1)Un = \frac{1}{2}n + \frac{3}{4}$$

$$Un = \frac{1}{\Delta(3 + 3\Delta + \Delta^2)}\left(\frac{1}{2}n + \frac{3}{4}\right)$$

$$Un = \frac{1}{3\left(1 + \Delta + \frac{1}{3}\Delta^2\right)}\left(\frac{1}{4}n^{(2)} + \frac{3}{4}n^{(1)}\right)$$

$$Un = \frac{1}{3}\left[1 - \left(\Delta + \frac{1}{3}\Delta^2\right) + \left(\Delta + \frac{1}{3}\Delta^2\right)^2 - \dots\right]\left(\frac{1}{4}n^{(2)} + \frac{3}{4}n^{(1)}\right)$$

$$Un = \frac{1}{12}n^2 - \frac{5}{36}$$

The general solution of *Un* is like this:

$$Un = \frac{1}{12}n^2 - \frac{5}{36} + A_4 + A_6(-1)^n + 2A_2\cos\frac{2}{3}n\pi$$

$$n = 3,\ n = 4,\ n = 5$$

$$A_4 - A_6 + 2A_2 = \frac{1}{4}$$

$$A_4 + A_6 - A_2 = -\frac{1}{3}$$

$$A_4 - A_6 - A_2 = -\frac{1}{12}$$

Thus

$$A_4 = -\frac{7}{72}A_6 = -\frac{1}{8}A_2 = \frac{1}{9}$$

Therefore, we have the general solution where k=3

$$Un = \frac{1}{12}n^2 - \frac{7}{72} - \frac{1}{8}(-1)^n + \frac{2}{9}\cos\frac{2}{3}n\pi$$

Here,

$$\frac{2}{9}\cos\frac{2n\pi}{3} = \frac{1}{9}\left(\cos\frac{2n\pi}{3} + \cos\frac{4n\pi}{3}\right)$$

(3)For K=4,

$$Un+4-Un = \frac{1}{12}n^2 - \frac{7}{72} - \frac{1}{8}(-1)^n + \frac{2}{9}\cos\frac{2}{3}n\pi + \frac{1}{2}n + \frac{1}{4}(-1)^n - \frac{1}{4} + 1$$
$$Un+4-Un = \frac{1}{12}n^2 + \frac{1}{2}n + \frac{1}{8}(-1)^n + \frac{2}{9}\cos\frac{2}{3}n\pi + \frac{1}{4}(-1)^n + \frac{47}{72}$$
$$\left(E^4-1\right)Un = \frac{1}{12}n^{(2)} + \frac{7}{12}n^{(1)} + \frac{47}{72}$$

Go though the same procedure, we have

$$Un = \frac{1}{16}\left(\frac{1}{9}n^3 + \frac{1}{3}n^2 - \frac{1}{2}n\right)$$

The general solution is like this:

$$Un = \frac{1}{16}\left(\frac{1}{9}n^3 + \frac{1}{3}n^2 - \frac{1}{2}n\right) + A_1 n(-1)^n + A_2 + A_3(-1)^n + A_4\sin\frac{2}{3}n\pi + A_6\cos\frac{2}{3}n\pi + A_5\cos\frac{n\pi}{2}$$

Here we have $n\cdot(-1)^n$, it can be explained because there are duplicated $(-1)^n$ as root.
There are two methods to calculate the constant *A*:
One is to use equations, as shown above when we do k=3

The second one is like below

For example:

$$A_1(-1)^n(n+4) - A_1(-1)^n n = \frac{1}{8}(-1)^n$$

then

$$A_1 = \frac{1}{32}$$

It is nearly impossible to know the results of

$$\frac{1}{E^k-1}\cos\frac{2n\pi}{3}$$

However we know cos’x=-sinxsin’x=cosx
Then, we know the result will be something like

$$A_4\sin\frac{2n\pi}{3} + A_6\cos\frac{2n\pi}{3}$$
.

So we can have this equation:

$$A_4\ \sin\frac{2}{3}(n+4)\pi + A_6\ \cos\frac{2}{3}(n+4)\pi - A_4\ \sin\frac{2}{3}n\pi - A_6\ \cos\frac{2}{3}n\pi = \frac{2}{9}\cos\frac{2}{3}n\pi$$

Then we have two equations:

$$-\frac{3}{2}A_4 - \frac{\sqrt{3}}{2}A_6 = 0$$
$$\frac{\sqrt{3}}{2}A_4 - \frac{3}{2}A_6 = \frac{2}{9}$$

Then

$$A_4 = \frac{1}{27}\sqrt{3}, \quad A_6 = -\frac{1}{9}$$

We need 3 equations to get the other 3 constant coefficients.

$$n = 0 \quad A_2 + A_5 + A_3 = \frac{1}{9}$$

$$n = 1 \quad A_2 - A_3 = -\frac{11}{144}$$

$$n = 2 \quad A_2 - A_5 + A_3 = -\frac{5}{36}$$

then:

$$A_5 = \frac{1}{8} \quad A_2 = -\frac{13}{288} \quad A_3 = \frac{1}{32}$$

$$Un = \frac{1}{16}\left(\frac{1}{9}n^3 + \frac{1}{3}n^2 - \frac{1}{2}n\right) + \frac{1}{32}n(-1)^n - \frac{13}{288} + \frac{1}{32}(-1)^n + \frac{1}{27}\sqrt{3}\sin\frac{2}{3}n\pi - \frac{1}{9}\cos\frac{2}{3}n\pi + \frac{1}{8}\cos\frac{n}{2}$$

also:

$$\frac{1}{16}\left(\cos\frac{n\pi}{2} + \cos\frac{3n\pi}{2}\right) = \frac{1}{8}\cos\frac{n\pi}{2}$$

Go through the similar procedure, we can get the expression for k=5

$$Un = \frac{1}{2880}\left(n^4 + 10n^3 + 10n^2 - 75n\right) - \frac{1}{64}n(-1)^n - \frac{1849}{86400} - \frac{5}{128}(-1)^n - \frac{1}{27}\sqrt{3}\sin\frac{2}{3}n\pi$$

$$-\frac{1}{27}\cos\frac{2}{3}n\pi - \frac{1}{16}\cos\frac{n\pi}{2} + \frac{1}{16}\sin\frac{n\pi}{2} + \frac{2}{25}\left(\cos\frac{2n\pi}{5} + \cos\frac{4n\pi}{5}\right)$$

Again, we know

$$\frac{1}{25}\left(\cos\frac{2n\pi}{5} + \cos\frac{4n\pi}{5} + \cos\frac{6n\pi}{5} + \cos\frac{8n\pi}{5}\right) = \frac{2}{25}\left(\cos\frac{2n\pi}{5} + \cos\frac{4n\pi}{5}\right)$$

4. How to Work on any K Partition Problem

When we review and compare the expressions for k=1, k=2, k=3, k=4 and k=5, we can certainly find some similarities:

Here are something we noticed:

1 The expressions are composed of the root of unity for each k with the constants.

2 The constant are 1, 1/4, 1/9, 1/16, 1/25 for

$$\cos 2n\pi, \ \cos n\pi, \ \cos\frac{2n\pi}{3}, \cos\frac{n\pi}{2}, \cos\frac{2n\pi}{5}$$

For k, we expect the constant for $\cos\frac{2n\pi}{k}$ is $1/k^2$, which is an interesting topic to prove it correct.

3 The expression for partitions k= 5 has to be concluded from partition k=4, partitions k=4 has to be concluded from partition k=3, ….partition k=3 from partition k=2, partition k=2 is from partition k=1

4 the constant for $1,\ n,\ n^2, n^3, n^4$ are shown as 1, 1/2 , 1/12, 1/144, 1/2880.. Then the constant for n^{k-1} will be $\frac{1}{k!(k-1)!}$. This can be easily proved. Because when we do $\frac{1}{E^k-1}n^{k-2}$, k*(k-1) will be part of the new constant.

The question now is for any k partition and number n , do we really have a solid expression for $U(n,k)$?
With our analysis, the $U(n,k)$ will be like this:

$$
\begin{aligned}
&\left(A_{1\,k-1}\cdot n^{k-1}+A_{1\,k-2}\cdot n^{k-2}+\ldots+A_{1\,1}\cdot n+A_{1\,0}\right)\cos 2n\pi+\\
&\left(A_{2\,\left[\frac{k}{2}-1\right]}\cdot n^{\left[\frac{k}{2}\right]-1}+A_{2\,\left[\frac{k}{2}-2\right]}\cdot n^{\left[\frac{k}{2}\right]-2}+\ldots+A_{2\,1}\cdot n+A_{2\,0}\right)\cos n\pi+\\
&\left(A_{3\,\left[\frac{k}{3}\right]-1}\cdot n^{\left[\frac{k}{3}\right]-1}+A_{3\,\left[\frac{k}{3}\right]-2}\cdot n^{\left[\frac{k}{3}\right]-2}+\ldots+A_{3\,1}\cdot n+A_{3\,0}\right)\cos\frac{2}{3}n\pi+\\
&\ldots+A_{k-1}\left(\cos\frac{2n\pi}{k-1}+\cos\frac{4n\pi}{k-1}+\ldots\cos\frac{2n\pi(k-2)}{k-1}\right)+\\
&\left(B_{1\,k-1}\cdot n^{k-1}+B_{1\,k-2}\cdot n^{k-2}+\ldots+B_{1\,1}\cdot n+B_{1\,0}\right)\sin 2n\pi+\\
&\left(B_{2\,\left[\frac{k}{2}-1\right]}\cdot n^{\left[\frac{k}{2}\right]-1}+B_{2\,\left[\frac{k}{2}-2\right]}\cdot n^{\left[\frac{k}{2}\right]-2}+\ldots+B_{2\,1}\cdot n+B_{2\,0}\right)\sin n\pi+\\
&\left(B_{3\,\left[\frac{k}{3}\right]-1}\cdot n^{\left[\frac{k}{3}\right]-1}+B_{3\,\left[\frac{k}{3}\right]-2}\cdot n^{\left[\frac{k}{3}\right]-2}+\ldots+B_{3\,1}\cdot n+B_{3\,0}\right)\sin\frac{2}{3}n\pi+\\
&..+B_{k-1}\left(\sin\frac{2n\pi}{k-1}+\sin\frac{4n\pi}{k-1}+\ldots\sin\frac{2n\pi(k-2)}{k-1}\right)+\\
&A_k\left(\cos\frac{2n\pi}{k}+\cos\frac{4n\pi}{k}+\ldots\cos\frac{2n\pi(k-1)}{k}\right)+A
\end{aligned}
$$

Here A, B is the constant for each variable, to get all the constant coefficients for each variable, we can use equations to solve, which has been addressed when we do k=2,3,4,5.

If we define P(n) to be all the ways of partitions of n, it has been an interesting topic to know the value of P(n)

Actually, since we have:
$U(n,k)=U(n-k,1)+U(n-k,2)+U(n-k,3)+\ldots+U(n-k,k)$
Then we can have
$U(2n,n)=P(n)$
This means as long as we know how to get value the U(n,k) we can also calculate the P(n)

To avoid the complicated computation, we can also consider to use $A_{k-1}n^{k-1}+A_{k-2}n^{k-2}+\ldots+A_1n$ to do the estimate for the $U(n,k)$ although it is not an accurate number.

5. CONCLUSION

As shown in this paper, a new method to analyze the integer partition problem has been introduced. We have got some interesting results when k=2, k=3, k=4 and k=5, which have been verified to be correct. Although the calculation will become more and more complex when n and k becomes larger, we can still be confident to claim there will be a formal expression existing to calculate the number of U(n,k), which is a big step in the journey of getting the final solution of the integer partition problem. The methods we use here are new and interesting, which can help other researchers to do more research in this field. When the calculation is too much, we can also use computers to do the job since we have already found the method to generate U(n,k) from U(n,k-1).

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